1. *Suppose $\{a_n\}_{n\geq 1}$ is a bounded sequence and $\{b_n\}_{n\geq 1}$ is a sequence converging to 0 as n tends to ∞. Show that $\{a_nb_n\}_{n\geq 1}$ conveerges to 0 as n tends to ∞.*

   Solution: Since $\{a_n\}_{n\geq 1}$ is bounded, there exists $M > 0$ such that $|a_n| \leq M$ for all $n \geq 1$. Let $\epsilon > 0$. Since $\{b_n\}_{n\geq 1}$ converges to 0, there exists $N \in \mathbb{N}$ such that

   $$|b_n| < \frac{\epsilon}{M} \text{ for all } n \geq N$$

   Now

   $$|a_nb_n| \leq M|b_n| < M\frac{\epsilon}{M} = \epsilon \text{ for all } n \geq N$$

   Hence $\{a_nb_n\}_{n\geq 1}$ conveerges to 0 as n tends to ∞. □

2. *Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers and $c = \liminf_{n\to\infty} x_n$. Show that for any $\epsilon > 0$ the set $M_\epsilon = \{n : x_n < c - \epsilon\}$ is a finite set.*

   Solution: Note that $c = \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k$. Suppose $M_\epsilon = \{n : x_n < c - \epsilon\}$ is an infinite set, then for any $n \in \mathbb{N}$ there exists a $k_n \in M_\epsilon$ such that $k_n > n$. This implies that $\inf_{k\geq n} x_k < c - \epsilon$. This implies that $c = \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k \leq c - \epsilon$. This is a contradiction as $\epsilon > 0$. Hence M_ϵ is a finite set. □

3. *Show that the polynomial $p(x) = 3x^5 + 2x^3 + 6x + 5$ has exactly one real root.*

   Solution: By the fundamental theorem of algebra p has all its roots in $\mathbb{C}$. Since p is an odd degree polynomial and since the complex roots of a real polynomial occur as conjugate pairs, p must have a real root.
   Note that $p'(x) = 15x^4 + 6x^2 + 6$ for all $x \in \mathbb{R}$ and $p'(x) > 0$ for all $x \in \mathbb{R}$. Therefore p is a strictly increasing function on $\mathbb{R}$. Hence it can have only one real root.
   Thus p has exactly one real root. □

4. *Let X be the set of finite subsets of $\mathbb{N}$. Show that X is countable.*

   Solution: Note that $\mathbb{N} \times \mathbb{N} = \cup_{n\in\mathbb{N}}\{n\} \times \mathbb{N}$, a countable union of countable sets, hence it is countable. Thyerefore by induction $\underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{\text{n-times}}$ is countable , for all n. Hence

   $$Y := \bigcup_{n=0}^{\infty} \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{\text{n-times}} \text{ is countable}$$

   where $n = 0$ corresponds to the empty set ϕ. Now define $\varphi : X \to Y$ as follows. Define $\varphi(\phi) = \phi$. Let $A = \{x_1, x_2, ..., x_n\} \in X$ be non-empty, where $x_i \in \mathbb{N}, x_1 < x_2 < ... < x_n$. Then d efine $\varphi(A) = (x_1, x_2, ..., x_n) \in Y$. Then clearly φ is injective. Hence X is countable.

5. *State and prove the mean value theorem (you may assume Rolle's theorem).*

Solution:
Staement: Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof: Copnsider the function φ defined on I by

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then φ is continuous on $[a, b]$, differentiable on (a, b) and $\varphi(a) = \varphi(b) = 0$. Therefore by Rolle's theorem, there exists a point c in (a, b) such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Hence, $f(b) - f(a) = f'(c)(b - a)$. $\square$

6. *Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions. Define h, k on $[0, 1]$ by $h(x) = \min\{f(x), g(x)\}$ and $k(x) = \max\{f(x), g(x)\}$. Show that h, k are continuous. Give examples to show that both h, k need not be differentiable, even if f, g are differentiable.*

Solution: Note that $h(x) = \frac{|f(x)-g(x)|+f(x)+g(x)}{2}, k(x) = -(\frac{|f(x)-g(x)|-f(x)-g(x)}{2})$. Now since $x \mapsto |x|$ is continuous and composition of continuous functions is continuous h, k are continuous.
Example: Let $f, g : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x - \frac{1}{2}, g(x) = \frac{1}{2} - x$ for all $x \in [0, 1]$. Then f, g are differentible on $[0, 1]$ and

$$h(x) = \begin{cases} \frac{1}{2} - x & \text{if } 0 \leq x \leq \frac{1}{2}; \\ x - \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \quad \text{and } k(x) = -h(x)$$

Both h and k are not differentiable at $\frac{1}{2}$. $\square$

7. *Suppose $a : \mathbb{R} \to \mathbb{R}$ is a function such that*

$$a(\frac{x + y}{2}) \leq \frac{a(x) + a(y)}{2}, \qquad x, y \in \mathbb{R}$$

   (i) Show that for all $n \geq 2$ and for all $x_1, x_2, ..., x_n$ in $\mathbb{R}$

$$a\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a(x_i)$$

   (ii) If a is continuous show that

$$a(px + (1 - p)y) \leq pa(x) + (1 - p)a(y)$$

   for all $x, y \in \mathbb{R}$ and $0 \leq p \leq 1$. (Hint: First prove the result for rational numbers p.)

Solution:

(i) We have to prove that for all $n \geq 2$ and for all $x_1, x_2, ..., x_n$ in $\mathbb{R}$

$$a\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a(x_i) \tag{*}$$

First we shall prove this all n of the form $n = 2^k, k \in \mathbb{N}$. This we prove by induction on k. Note that (*) is true for $k = 1$ by hypothesis. Assume that (*) is true for all $n = 2^k, k = 1, 2, ..., m$. Let $x_1, x_2, ..., x_{2^{m+1}} \in \mathbb{R}$, then

$$\begin{aligned}
a\left(\frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}} x_i\right) &= a\left(\frac{1}{2}\left(\frac{1}{2^m}\sum_{i=1}^{2^m} x_i + \frac{1}{2^m}\sum_{i=1}^{2^m} x_{2^m+i}\right)\right) \\
&\leq \frac{1}{2}a\left(\frac{1}{2^m}\sum_{i=1}^{2^m} x_i\right) + \frac{1}{2}a\left(\frac{1}{2^m}\sum_{i=1}^{2^m} x_{2^m+i}\right) \text{ (by hypothesis)} \\
&\leq \frac{1}{2}\left(\frac{1}{2^m}\sum_{i=1}^{2^m} a(x_i)\right) + \frac{1}{2}\left(\frac{1}{2^m}\sum_{i=1}^{2^m} a(x_{2^m+i})\right) \text{ (by induction hypothesis)} \\
&= \frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}} a(x_i)
\end{aligned}$$

This proves it for all $n = 2^k, k \in \mathbb{N}$.
Now let $n \in \mathbb{N}, 2^k < n < 2^{k+1}$. Let $y := \frac{1}{n}\sum_{i=1}^{n} x_i, m := 2^{k+1} - n$. Consider $a(y) = a(\frac{1}{2^{k+1}}(x_1 + x_2 + ... + x_n + m \cdot y)) \leq \frac{1}{2^{k+1}}\left(\sum_{i=1}^{n} a(x_i) + ma(y)\right)$, hence $(2^{k+1} - m)a(y) \leq \sum_{i=1}^{n} a(x_i)$. That is $a(y) \leq \frac{1}{n}\sum_{i=1}^{n} a(x_i)$. This completes the proof.

(ii) Let $p = \frac{m}{n}, m < n$. Then $1 - p = \frac{n-m}{n}$ and

$$\begin{aligned}
a(px + (1-p)y) &= a(\frac{mx + (n-m)y}{n}) \\
&\leq \frac{1}{n}(ma(x) + (n-m)a(y)) \text{ from (i)} \\
&= pa(x) + (1-p)a(y).
\end{aligned}$$

Hence (ii) is true for all rational $p, 0 < p < 1$. Now since the set $\{p \in \mathbb{Q} : 0 < p < 1\}$ is dense in $[0, 1]$ and a is contiuous (ii) holds for all $0 \leq p \leq 1$.